On the waterbag model of the dispersionless KP hierarchy (II)

2007 J. Phys. A: Math. Theor. 40 12973

(http://iopscience.iop.org/1751-8121/40/43/009)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 12973–12985 doi:10.1088/1751-8113/40/43/009

On the waterbag model of the dispersionless KP hierarchy (II)

Jen-Hsu Chang

Department of Computer Science, National Defense University, Tauyuan, Taiwan

E-mail: jhchang@ccit.edu.tw

Received 13 August 2007, in final form 10 September 2007
Published 9 October 2007
Online at stacks.iop.org/JPhysA/40/12973

Abstract
We construct the bi-Hamiltonian structure of the waterbag model of dKP and establish the third-order Hamiltonian operator associated with the waterbag model. Also, the symmetries and conserved densities of the rational type are discussed.

PACS numbers: 02.30.Ik, 02.30.Jr
Mathematics Subject Classification: 35Q58, 37K10, 37K35

1. Introduction

The dispersionless KP hierarchy(dKP or Benney moment chain) is defined by

$$\partial_{t_n}\lambda(z) = \{\lambda(z), B_n(z)\}, \qquad n = 1, 2, \ldots, \tag{1}$$

where the Lax operator $\lambda(z)$ is

$$\lambda(z) = z + \sum_{1}^{\infty} v_{n+1} z^{-n} \tag{2}$$

and

$$B_n(z) = \frac{[\lambda^n(z)]_+}{n}, \qquad i = 1, 2, 3, \ldots \quad t_1 = x.$$

Here $[\cdots]_+$ denotes the non-negative part of the Laurent series $\lambda^n(z)$. For example,

$$B_2 = \frac{z^2}{2} + v_2, \qquad B_3 = \frac{z^3}{3} + v_2 z + v_3.$$

Finally, the bracket in (1) denotes the natural Poisson bracket on the space of functions of the two variables (x, z):

$$\{f(x, z), g(x, z)\} = \partial_x f \partial_z g - \partial_x g \partial_z f. \tag{3}$$

The compatibility of (1) coincides with the zero-curvature equation

$$\partial_m B_n(z) - \partial_n B_m(z) = \{B_n(z), B_m(z)\}. \tag{4}$$

If we denote $t_2 = y$ and $t_3 = t$, then equation (4) for $m = 2, n = 3$ gives

$$v_{3x} = v_{2y} \qquad v_{3y} = v_{2t} - v_2 v_{2x},$$

from which the dKP equation is derived ($v_2 = v$):

$$v_{yy} = (v_t - v v_x)_x. \tag{5}$$

According to the dKP theory [1, 11, 13, 26], there exists a wavefunction $S(\lambda, x, t_2, t_3, \ldots)$ such that $z = S_x$ and satisfies the Hamiltonian–Jacobian equation

$$\frac{\partial S}{\partial t_n} = B_n(z)|_{z=S_x}. \tag{6}$$

It can be seen that the compatibility of (6) also implies the zero-curvature equation (4). Now, we expand $B_n(z)$ as

$$B_n(z(\lambda)) = \frac{[\lambda^n(z)]_+}{n} = \frac{\lambda^n}{n} - \sum_{i=0}^{\infty} G_{in}\lambda^{-i-1},$$

where the coefficients can be calculated by the residue form

$$G_{in} = -\mathrm{res}_{\lambda=\infty}(\lambda^i B_n(z)\,\mathrm{d}\lambda) = \frac{1}{i+1}\mathrm{res}_{z=\infty}\left(\lambda^{i+1}\frac{\partial B_n(z)}{\partial z}\,\mathrm{d}z\right),$$

from which the symmetry property

$$G_{in} = G_{ni}$$

can be easily deduced. Moreover [26], it can be shown that the polynomials B_n must satisfy the integrability condition

$$\frac{\partial B_m(\lambda)}{\partial t_n} = \frac{\partial B_n(\lambda)}{\partial t_m},$$

from which in turns follows the integrability of the coefficients G_{in}, i.e., there exists the free energy $\mathcal{F}$ (dispersionless τ function) such that

$$G_{in} = \frac{\partial^2 \mathcal{F}}{\partial t_i \partial t_n}.$$

This latter function may be for example used to invert formula (2) :

$$z = \lambda - \frac{\mathcal{F}_{11}}{\lambda} - \frac{\mathcal{F}_{12}}{2\lambda^2} - \frac{\mathcal{F}_{13}}{3\lambda^3} - \frac{\mathcal{F}_{14}}{4\lambda^4} - \cdots, \tag{7}$$

where $\mathcal{F}_{1n}$ are polynomials of $v_2, v_3, \ldots, v_{n+1}$ and in fact

$$h_n \equiv \frac{\mathcal{F}_{1n}}{n} = \mathrm{res}_{z=\infty}\frac{\lambda^n}{n}\,\mathrm{d}z \tag{8}$$

are the conserved densities for the dKP hierarchy (1). In [3, 4], it is proved that the dKP hierarchy (1) is equivalent to the dispersionless Hirota equation

$$D(\lambda)S(\lambda') = -\log\frac{z(\lambda) - z(\lambda')}{\lambda}, \tag{9}$$

where $D(\lambda)$ is the operator $\sum_{n=1}^{\infty}\frac{1}{n\lambda^n}\frac{\partial}{\partial t_n}$.

Next, we consider the symmetry constraint [2]

$$\mathcal{F}_x = \sum_{i=1}^{M}\epsilon_i S_i, \tag{10}$$

where $S_i = S(\lambda_i)$, λ_i are points in the complex plane and ϵ_i are constants. Note that from (7) we know

$$D(\lambda)\mathcal{F}_x = \lambda - z.$$

On the other hand, by (9) and (10), we also have

$$\begin{aligned} D(\lambda)\mathcal{F}_x &= \sum_{i=1}^{M} \epsilon_i D(\lambda) S_i = -\sum_{i=1}^{M} \epsilon_i \ln \frac{z(\lambda) - z(\lambda_i)}{\lambda} \\ &= -\sum_{i=1}^{M} \epsilon_i \ln(z - p^i) + \left(\sum_{i=1}^{M} \epsilon_i\right) \ln \lambda, \end{aligned}$$

where $z = z(\lambda)$ and $p^i = z(\lambda_i)$. We assume that

$$\sum_{i=1}^{M} \epsilon_i = 0$$

and then we get the waterbag reduction [2, 18]

$$\lambda = z - \sum_{i=1}^{M} \epsilon_i \ln(z - p^i) \tag{11}$$

$$= z + \sum_{n=1}^{\infty} \frac{v_{n+1}}{z^n}, \tag{12}$$

where $v_{n+1} = \frac{1}{n} \sum_{i=1}^{M} \epsilon_i (p^i)^n$. From this we obtain

$$B_2(z) = \frac{1}{2} z^2 + \sum_{i=1}^{M} \epsilon_i p^i.$$

So $(t_2 = y)$

$$\partial_y p^i = \partial_x \left[\frac{1}{2} (p^i)^2 + \sum_{i=1}^{M} \epsilon_i p^i \right]. \tag{13}$$

In [5], the two-component case of (13) is investigated. We see that equation (13) can be written as the Hamiltonian system

$$\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_y = \frac{1}{2} \begin{bmatrix} \frac{1}{\epsilon_1} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{\epsilon_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\epsilon_M} \end{bmatrix} \partial_x \begin{bmatrix} \frac{\delta H_3}{\delta p^1} \\ \frac{\delta H_3}{\delta p^2} \\ \vdots \\ \frac{\delta H_3}{\delta p^M} \end{bmatrix},$$

where δ is the variation derivative and

$$\begin{aligned} H_3 &= \frac{1}{3} \int \mathrm{d}x\, \mathrm{Res}(\lambda^3\, \mathrm{d}z) = \int \mathrm{d}x \left(v_2^2 + v_4\right) \\ &= \int \mathrm{d}x \left[\left(\sum_{i=1}^{M} \epsilon_i p^i\right)^2 + \frac{1}{3} \left(\sum_{i=1}^{M} \epsilon_i (p^i)^3\right) \right]. \end{aligned}$$

A bi-Hamiltonian structure is defined as (in the case of dKP)

$$\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_y = -\frac{1}{2} J_1 \begin{bmatrix} \frac{\delta H_3}{\delta p^1} \\ \frac{\delta H_3}{\delta p^2} \\ \vdots \\ \frac{\delta H_3}{\delta p^M} \end{bmatrix} = J_2 \begin{bmatrix} \frac{\delta H}{\delta p^1} \\ \frac{\delta H}{\delta p^2} \\ \vdots \\ \frac{\delta H}{\delta p^M} \end{bmatrix},$$

where

$$J_1 = -\begin{bmatrix} \frac{1}{\epsilon_1} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{\epsilon_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\epsilon_M} \end{bmatrix} \partial_x$$

and J_2 is also a Hamiltonian operator which is compatible with J_1, i.e., $J_1 + cJ_2$ is also a Hamiltonian one for any complex number c [8, 14, 20]. We hope to find J_2 and the related Hamiltonian H.

Furthermore, from the bi-Hamiltonian structure (24) or (26) (see below) of the waterbag model (11), we also find the recursion operator $\hat{R}$ in (25) (see below) is local. Then, according to the bi-Hamiltonian theory [14, 22], one can construct rational symmetries using the local recursion operator $\hat{R}$(equation (25)). Hence the higher-order rational conserved densities (quasi-rational functions) are investigated.

This paper is organized as follows. In the next section, we construct the bi-Hamiltonian structure of the waterbag model from the Landau–Ginsburg formulation in topological field theory. Section 3 is devoted to investigating the quasi-rational symmetries and the corresponding conserved densities of the waterbag model. In the final section, one discusses some problems to be investigated

2. Free energy and the bi-Hamiltonian structure

In this section, we investigate the relations between the bi-Hamiltonian structure and the free energy of the waterbag model.

The free energy is a function $\mathbb{F}(t^1, t^2, \dots, t^n)$ such that the associated functions,

$$c_{ijk} = \frac{\partial^3 \mathbb{F}}{\partial t^i \partial t^j \partial t^k},$$

satisfy the following conditions.

- The matrix $\eta_{ij} = c_{1ij}$ is constant and non-degenerate. This, together with the inverse matrix η^{ij}, is used to raise and lower indices.
- The functions $c^i_{jk} = \eta^{ir} c_{rjk}$ define an associative commutative algebra with a unity element(Frobenius algebra).

Equations of associativity give a system of nonlinear PDE for $\mathbb{F}(t)$:

$$\frac{\partial^3 \mathbb{F}(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 \mathbb{F}(t)}{\partial t^\mu \partial t^\gamma \partial t^\sigma} = \frac{\partial^3 \mathbb{F}(t)}{\partial t^\alpha \partial t^\gamma \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 \mathbb{F}(t)}{\partial t^\mu \partial t^\beta \partial t^\sigma}.$$

These equations constitute the Witten–Dijkgraaf–Verlinde–Verlinde (or WDVV) equations. The geometrical setting in which to understand the free energy $\mathbb{F}(t)$ is the Frobenius manifold

[8, 9]. One way to construct such manifold is derived via Landau–Ginzburg formalism as the structure on the parameter space M of the appropriate form

$$\lambda = \lambda(z; t^1, t^2, \ldots, t^n).$$

The Frobenius structure is given by the flat metric

$$\eta(\partial, \partial') = -\sum \mathrm{res}_{\mathrm{d}\lambda=0} \left\{ \frac{\partial(\lambda\,\mathrm{d}z)\partial'(\lambda\,\mathrm{d}z)}{\mathrm{d}\lambda(z)} \right\}$$

and the tensor

$$c(\partial, \partial', \partial'') = -\sum \mathrm{res}_{\mathrm{d}\lambda=0} \left\{ \frac{\partial(\lambda\,\mathrm{d}z)\partial'(\lambda\,\mathrm{d}z)\partial''(\lambda\,\mathrm{d}z)}{\mathrm{d}\lambda(z)\,\mathrm{d}z} \right\}$$

defines a totally symmetric (3, 0)-tensor c_{ijk}.

Geometrically, a solution of WDVV equation defines a multiplication

$$\circ : TM \times TM \longrightarrow TM$$

of vector fields on the parameter space M, i.e,

$$\partial_{t^\alpha} \circ \partial_{t^\beta} = c^\gamma_{\alpha\beta}(t)\partial_{t^\gamma}.$$

From $c^\gamma_{\alpha\beta}(t)$, one can construct integrable hierarchies whose corresponding Hamiltonian densities are defined recursively by the formula

$$\frac{\partial^2 \psi^{(l)}_\alpha}{\partial t^i \partial t^j} = c^k_{ij} \frac{\partial \psi^{(l-1)}_\alpha}{\partial t^k}, \tag{14}$$

where $l \geqslant 1, \alpha = 1, 2, \ldots, n$, and $\psi^0_\alpha = \eta_{\alpha\epsilon} t^\epsilon$. The integrability conditions for this systems are automatically satisfied when the c^k_{ij} are defined as above.

For the waterbag model (11), we have the following

Theorem 2.1. *[10]:*

$$\eta\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}\right) = \eta_{ij} = -\epsilon_i \delta_{i,j} \quad i, j = 1, 2, \ldots M,$$

$$c\left(\frac{\partial}{\partial p^\alpha}, \frac{\partial}{\partial p^\beta}, \frac{\partial}{\partial p^\gamma}\right) = c_{\alpha\beta\gamma} = 0, \quad \alpha, \quad \beta, \quad \gamma \quad \text{distinct},$$

$$c\left(\frac{\partial}{\partial p^\alpha}, \frac{\partial}{\partial p^\alpha}, \frac{\partial}{\partial p^\beta}\right) = c_{\alpha\alpha\beta} = \frac{\epsilon_\alpha \epsilon_\alpha}{p^\beta - p^\alpha}, \qquad \alpha \neq \beta,$$

$$c\left(\frac{\partial}{\partial p^\alpha}, \frac{\partial}{\partial p^\alpha}, \frac{\partial}{\partial p^\alpha}\right) = c_{\alpha\alpha\alpha} = -\epsilon_\alpha + \sum_{r\neq\alpha} \frac{\epsilon_\alpha \epsilon_r}{p^\alpha - p^r}, \qquad \alpha \neq \beta.$$

Let us define

$$\Omega = \sum_{i=1}^M \frac{\partial}{\partial p^i}. \tag{15}$$

Then we can see that

$$\eta\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}\right) = c\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}, \Omega\right). \tag{16}$$

From the theorem, one can get the free energy associated with the waterbag model (11), noting that p^i are flat coordinates,

$$\mathbb{F}(\vec{p}) = -\frac{1}{6}\sum_{k=1}^M \epsilon_k (p^k)^3 + \frac{1}{8}\sum_{i\neq j} \epsilon_i \epsilon_j (p^i - p^j)^2 \ln(p^i - p^j)^2, \tag{17}$$

where $\vec{p} = (p^1, p^2, \ldots, p^M)$ and from (16) one has

$$t^1 = \sum_{i=1}^{M} p^i.$$

Also, we have

$$\begin{aligned} c^{\alpha}_{\beta\gamma} &= 0, \quad \alpha, \beta, \gamma \quad \text{distinct}, \\ c^{\beta}_{\alpha\alpha} &= \frac{\epsilon_\alpha}{p^\alpha - p^\beta}, \qquad \alpha \neq \beta \\ c^{\alpha}_{\alpha\beta} &= \frac{\epsilon_\beta}{p^\alpha - p^\beta}, \qquad \alpha \neq \beta \\ c^{\alpha}_{\alpha\alpha} &= 1 - \sum_{r\neq\alpha} \frac{\epsilon_r}{p^\alpha - p^r}. \end{aligned}$$

If we define $e_i = \frac{\partial\lambda}{\partial p^i} = \frac{\epsilon_i}{z-p^i}$, then it is not difficult to see that

$$e_i e_j = c^k_{ij} e_k + Q_{ij}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}z}\right), \tag{18}$$

where

$$Q_{ij} = \begin{cases} -\dfrac{\epsilon_i}{z-p^i}, & i = j, \\ 0, & i \neq j. \end{cases}$$

Hence we have the recursion relation

$$\begin{aligned} \frac{\partial^2 h_n}{\partial p^i \partial p^j} &= \frac{\partial}{\partial p^i} \oint_\infty \lambda^{n-1} \frac{\partial\lambda}{\partial p^j}\,\mathrm{d}z = \frac{\partial}{\partial p^i} \oint_\infty \lambda^{n-1} \frac{\epsilon_j}{z-p^j}\,\mathrm{d}z \\ &= (n-1) \oint_\infty \lambda^{n-2} \frac{\epsilon_i}{z-p^i}\frac{\epsilon_j}{z-p^j}\,\mathrm{d}z + \oint_\infty \lambda^{n-1} \frac{\partial}{\partial p^i}\left(\frac{\epsilon_j}{z-p^j}\right)\mathrm{d}z \\ &= (n-1) \oint_\infty \lambda^{n-2}\left(c^k_{ij} e_k + Q_{ij}\right)\frac{\mathrm{d}\lambda}{\mathrm{d}z} - \oint_\infty \lambda^{n-1}\frac{\partial}{\partial z}\left(\frac{\epsilon_j}{z-p^j}\right)\mathrm{d}z \\ &= c^k_{ij}(n-1) \oint_\infty \lambda^{n-2} \frac{\epsilon_k}{z-p^k}\,\mathrm{d}z - \oint_\infty \frac{\partial}{\partial z}\left(\lambda^{n-1}\frac{\epsilon_j}{z-p^j}\right)\mathrm{d}z \\ &= c^k_{ij} \frac{\partial}{\partial p^k} \oint_\infty \lambda^{n-1}\,\mathrm{d}z \\ &= (n-1) c^k_{ij} \frac{\partial h_{n-1}}{\partial p^k}, \qquad n \geqslant 1. \end{aligned} \tag{19}$$

Moreover, we also have

$$\begin{aligned} \left(\sum_i \partial_{p^i}\right) h_n &= \oint_\infty \lambda^{n-1}\left(\sum_{i=1}^{M}\frac{\epsilon_i}{z-p^i}\right)\mathrm{d}z \\ &= \oint_\infty \lambda^{n-1}\left(1 - \frac{\mathrm{d}\lambda}{\mathrm{d}z}\right) = (n-1)h_{n-1}. \end{aligned} \tag{20}$$

We can express (19) as

$$\frac{\partial h_n}{\partial p^i} = (n-1)\eta^{kl}\partial_x^{-1}\left[\left(\frac{\partial^2 \mathbb{F}}{\partial p^i \partial p^l}\right)_x \frac{\partial h_{n-1}}{\partial p^k}\right],$$

where

$$\eta^{kl} = -\begin{bmatrix} \frac{1}{\epsilon_1} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{\epsilon_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\epsilon_M} \end{bmatrix}.$$

Therefore we obtain the recursion operator [19]

$$R_i^k = \eta^{kl}\partial_x^{-1}\left[\left(\frac{\partial^2\mathbb{F}}{\partial p^i\partial p^l}\right)_x\right] = \partial_x^{-1}\left[\eta^{kl}\left(\frac{\partial^2\mathbb{F}}{\partial p^i\partial p^l}\right)_x\right]$$
$$= \partial_x^{-1}W_{ix}^k = \partial_x^{-1}\eta^{kl}\frac{\partial^3\mathbb{F}}{\partial p^i\partial p^l\partial p^\varepsilon}p_x^\varepsilon, \tag{21}$$

where

$$W_i^k = \eta^{kl}\frac{\partial^2\mathbb{F}}{\partial p^i\partial p^l}$$
$$= \begin{bmatrix} p^1 - \sum_{l\neq 1}\epsilon_l\ln(p^1-p^l) & \epsilon_2\ln(p^1-p^2) & \dots & \epsilon_M\ln(p^1-p^M) \\ \epsilon_1\ln(p^2-p^1) & p^2-\sum_{l\neq 2}\epsilon_l\ln(p^2-p^l) & \dots & \epsilon_M\ln(p^2-p^M) \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon_1\ln(p^M-p^1) & \epsilon_2\ln(p^M-p^2) & \dots & p^M-\sum_{l\neq M}\epsilon_l\ln(p^M-p^l) \end{bmatrix}. \tag{22}$$

Then

$$\begin{bmatrix} \frac{\partial h_n}{\partial p^1} \\ \frac{\partial h_n}{\partial p^2} \\ \vdots \\ \frac{\partial h_n}{\partial p^M} \end{bmatrix} = (n-1)(n-2)R^2\begin{bmatrix} \frac{\partial h_{n-2}}{\partial p^1} \\ \frac{\partial h_{n-2}}{\partial p^2} \\ \vdots \\ \frac{\partial h_{n-2}}{\partial p^M} \end{bmatrix}. \tag{23}$$

Also, it is known that the Hamiltonian operators [16, 17] (see also [19])

$$J_1 = \eta^{ij}\partial_x$$
$$J_2 = \sum_{m=1}^{M}\sum_{\alpha=1}^{M}\eta^{m\alpha}\eta^{i\epsilon}\eta^{jr}\frac{\partial^3\mathbb{F}}{\partial p^\epsilon\partial p^m\partial p^k}p_x^k\partial_x^{-1}\frac{\partial^3\mathbb{F}}{\partial p^\alpha\partial p^r\partial p^s}p_x^s$$
$$= \sum_{m=1}^{M}\sum_{\alpha=1}^{M}\left(W_m^i\right)_x\partial_x^{-1}\left(W_\alpha^j\right)_x$$

are compatible. Consequently, using (23) or $R^2 = J_1^{-1}J_2$, we obtain the bi-Hamiltonian structure for the waterbag model (11), $n \geqslant 2$,

$$\partial_{t_n}p^l = \frac{-1}{n}\eta^{li}\partial_x\frac{\partial h_{n+1}}{\partial p^i}$$
$$= -(n-1)\eta^{m\alpha}\left(W_m^l\right)_x\partial_x^{-1}\left(W_\alpha^i\right)_x\frac{\partial h_{n-1}}{\partial p^i}, \tag{24}$$

where W is defined by (22).

For $n = 2$, i.e.(13), one can directly verify the bi-Hamiltonian structure (24).

Remark. Using the Legendre-type transformation for WDVV equation in [8], we can introduce new flat coordinates from the first row vector of (22):

$$a_1 = \frac{1}{\epsilon_1}\left[p^1 - \sum_{l\neq 1}\epsilon_l \ln(p^1 - p^l)\right]$$
$$a_k = \ln(p^1 - p^k), \qquad k = 2, 3, 4, \ldots, M.$$

The inverse transformation of the above equation is

$$p^1 = \sum_{i=1}^{M}\epsilon_i a_i$$
$$p^k = \sum_{i=1}^{M}\epsilon_i a_i - \mathrm{e}^{a_k}, \qquad k = 2, 3, 4, \ldots, M.$$

Then the new free energy satisfying the WDVV equation is [18]

$$F = \frac{\epsilon_1^2 (a_1)^3}{6} + \frac{\epsilon_1 a_1}{2}\sum_{m\neq 1}\epsilon_m (a_m)^2 + P_3(\vec{a}) - \sum_{m\neq 1}\epsilon_m \mathrm{e}^{a_m}$$
$$+\frac{1}{2}\sum_{1<m<k}\epsilon_m\epsilon_k[Li_3(\mathrm{e}^{a_k-a_m}) + Li_3(\mathrm{e}^{a_m-a_k})],$$

where

$$P_3(\vec{a}) = \sum_{m\neq 1}\frac{\epsilon_m(\epsilon_m-\epsilon_1)\left(a_m^3\right)}{6} + \sum_{1<m<k}\frac{\epsilon_m\epsilon_k}{12}\left[(a_k+a_m)^3 - 2\left(a_k^3 + a_m^3\right)\right]$$

and $Li_3(\mathrm{e}^x)$ is defined by

$$Li_3(e^x) = \sum_{k=1}^{\infty}\frac{\mathrm{e}^{kx}}{k^3},$$

which has the properties

$$Li_3''(\mathrm{e}^x) = -\ln(1-\mathrm{e}^x), \qquad Li_3'''(\mathrm{e}^x) = \coth x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}}.$$

Using the different row vectors of (22), we can obtain different flat coordinate systems and then get different free energies using the Legendre transformation.

3. Higher-order symmetries and conservational laws

In this section, we investigate the symmetries and the conserved densities of the waterbag model involving the quasi-rational function. Quasi-rational means rational with respect to higher derivatives . This will generalize the results in [23].

We start with the recursion operator (21). It can be seen that

$$\hat{R} = J_1 R^{-1} J_1^{-1} = \partial_x (W_x)^{-1} \tag{25}$$

is the Sheftel–Teshkov recursion operator [19]. The bi-Hamiltonian structure (24) can also be written as ($n \geqslant 2$)

$$\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_{t_n} = \frac{(-1)}{n} J_1 \begin{bmatrix} \frac{\partial h_{n+1}}{\partial p^1} \\ \frac{\partial h_{n+1}}{\partial p^2} \\ \vdots \\ \frac{\partial h_{n+1}}{\partial p^M} \end{bmatrix} = \frac{(-1)}{n(n+1)(n+2)} \hat{J}_2 \begin{bmatrix} \frac{\partial h_{n+3}}{\partial p^1} \\ \frac{\partial h_{n+3}}{\partial p^2} \\ \vdots \\ \frac{\partial h_{n+3}}{\partial p^M} \end{bmatrix}, \tag{26}$$

the Hamiltonian operator $\hat{J}_2 = \hat{R}^2 J_1$ being third order. Also,

$$\hat{R}\begin{bmatrix} \frac{\partial h_n}{\partial p^1} \\ \frac{\partial h_n}{\partial p^2} \\ \vdots \\ \frac{\partial h_n}{\partial p^M} \end{bmatrix} = (n-1)\begin{bmatrix} \frac{\partial h_{n-1}}{\partial p^1} \\ \frac{\partial h_{n-1}}{\partial p^2} \\ \vdots \\ \frac{\partial h_{n-1}}{\partial p^M} \end{bmatrix}. \tag{27}$$

Next, we can express (13) as

$$\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_y = \mathbb{H}\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_x, \tag{28}$$

where

$$\mathbb{H} = \begin{bmatrix} p^1+\epsilon_1 & \epsilon_2 & \epsilon_3 & \dots & \epsilon_M \\ \epsilon_1 & p^2+\epsilon_2 & \epsilon_3 & \dots & \epsilon_M \\ \epsilon_1 & \epsilon_2 & p^3+\epsilon_3 & \dots & \epsilon_M \\ \vdots & \vdots & \vdots & \vdots & \epsilon_M \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \dots & p^M+\epsilon_M \end{bmatrix}.$$

We notice that $(\partial_y - \partial_x\mathbb{H})$ is Frechet's derivative operator for the system (28). It is not difficult to see that if

$$\partial_\tau \vec{p} = \vec{\mathbb{Q}}(\vec{p}, \vec{p}_x, \vec{p}_{xx}, \dots)$$

is commuting flow with (28), then we have

$$\partial_y \vec{\mathbb{Q}} = \partial_x(\mathbb{H}\vec{\mathbb{Q}}). \tag{29}$$

Lemma 3.1. *Let W be defined in (22). Then*

(1) $\partial_y W = \mathbb{H}\partial_x W$*;*
(2) $\mathbb{H}\partial_x W = (\partial_x W)\mathbb{H}$.

Proof. Direct computations. □

Theorem 3.2. *The recursion operator $\hat{R}$ satisfies the Lax representation*

$$\frac{\partial \hat{R}}{\partial y} = [\partial_x\mathbb{H}, \hat{R}]. \tag{30}$$

Proof. Using the lemma, we get

$$\begin{aligned} \frac{\partial \hat{R}}{\partial y} &= \partial_x \frac{\partial (W_x)^{-1}}{\partial y} = -\partial_x[(W_x)^{-1}W_{xy}(W_x)^{-1}] \\ &= -\partial_x(W_x)^{-1}[\mathbb{H}_x W_x + \mathbb{H}W_{xx}](W_x)^{-1} \\ &= -\partial_x(W_x)^{-1}\mathbb{H}_x - \partial_x(W_x)^{-1}\mathbb{H}W_{xx}(W_x)^{-1} \\ &= -\partial_x(W_x)^{-1}\mathbb{H}_x - \partial_x\mathbb{H}(W_x)^{-1}W_{xx}(W_x)^{-1} \\ &= -\partial_x(W_x)^{-1}\mathbb{H}_x - \partial_x\mathbb{H}[-\partial_x(W_x)^{-1} + (W_x)^{-1}\partial_x] \\ &= -\partial_x(W_x)^{-1}\mathbb{H}_x + \partial_x\mathbb{H}\partial_x(W_x)^{-1} - \partial_x\mathbb{H}(W_x)^{-1}\partial_x \\ &= -\partial_x(W_x)^{-1}\mathbb{H}_x + \partial_x\mathbb{H}\partial_x(W_x)^{-1} - \partial_x(W_x)^{-1}\mathbb{H}\partial_x \\ &= \partial_x\mathbb{H}\partial_x(W_x)^{-1} - \partial_x(W_x)^{-1}\partial_x\mathbb{H} \\ &= [\partial_x\mathbb{H}, \partial_x(W_x)^{-1}] = [\partial_x\mathbb{H}, \hat{R}]. \end{aligned}$$

□

Hence from the theorem one knows that Frechet's derivative operator and the recursion operator commute, i.e.,

$$(\partial_y - \partial_x \mathbb{H})\hat{R} = \hat{R}(\partial_y - \partial_x \mathbb{H}).$$

Moreover, if we let ($\mathbb{H}_2 = \mathbb{H}$)

$$\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_{t_n} = \frac{(-1)}{n} J_1 \begin{bmatrix} \frac{\partial h_{n+1}}{\partial p^1} \\ \frac{\partial h_{n+1}}{\partial p^2} \\ \vdots \\ \frac{\partial h_{n+1}}{\partial p^M} \end{bmatrix} = \mathbb{H}_n \begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_x,$$

where

$$\mathbb{H}_n = \frac{1}{n}\begin{bmatrix} \frac{1}{\epsilon_1}\frac{\partial^2 h_{n+1}}{\partial p^1 \partial p^1} & \frac{1}{\epsilon_1}\frac{\partial^2 h_{n+1}}{\partial p^1 \partial p^2} & \frac{1}{\epsilon_1}\frac{\partial^2 h_{n+1}}{\partial p^1 \partial p^3} & \cdots & \frac{1}{\epsilon_1}\frac{\partial^2 h_{n+1}}{\partial p^1 \partial p^M} \\ \frac{1}{\epsilon_2}\frac{\partial^2 h_{n+1}}{\partial p^2 \partial p^1} & \frac{1}{\epsilon_2}\frac{\partial^2 h_{n+1}}{\partial p^2 \partial p^2} & \frac{1}{\epsilon_2}\frac{\partial^2 h_{n+1}}{\partial p^2 \partial p^3} & \cdots & \frac{1}{\epsilon_2}\frac{\partial^2 h_{n+1}}{\partial p^2 \partial p^M} \\ \frac{1}{\epsilon_3}\frac{\partial^2 h_{n+1}}{\partial p^3 \partial p^1} & \frac{1}{\epsilon_3}\frac{\partial^2 h_{n+1}}{\partial p^3 \partial p^2} & \frac{1}{\epsilon_3}\frac{\partial^2 h_{n+1}}{\partial p^3 \partial p^3} & \cdots & \frac{1}{\epsilon_3}\frac{\partial^2 h_{n+1}}{\partial p^3 \partial p^M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\epsilon_M}\frac{\partial^2 h_{n+1}}{\partial p^M \partial p^1} & \frac{1}{\epsilon_M}\frac{\partial^2 h_{n+1}}{\partial p^M \partial p^2} & \frac{1}{\epsilon_M}\frac{\partial^2 h_{n+1}}{\partial p^M \partial p^3} & \cdots & \frac{1}{\epsilon_M}\frac{\partial^2 h_{n+1}}{\partial p^M \partial p^M} \end{bmatrix},$$

then from (27) we also obtain

$$\left(\partial_{t_n} - \partial_x \mathbb{H}_n\right)\hat{R} = \hat{R}\left(\partial_{t_n} - \partial_x \mathbb{H}_n\right). \tag{31}$$

Therefore we obtain the following

Theorem 3.3. *Let h_{n+1} be defined by (8) and $\mathbb{Q}_m$ be the flows defined by*

$$\vec{p}_{\tau_m} = \begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_{\tau_m} = \mathbb{Q}_m = \hat{R}^m \begin{bmatrix} xp_x^1 \\ xp_x^2 \\ \vdots \\ xp_x^M \end{bmatrix} = \hat{R}^{m-1}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Then $\mathbb{Q}_m$ is a commuting flow with $\vec{p}_{t_n}$, i.e., $\vec{p}_{t_n t_{\tau_m}} = \vec{p}_{t_{\tau_m} t_n}$ provided $m \geqslant n$.

Proof. Let us denote

$$x\vec{p}_x = \begin{bmatrix} xp_x^1 \\ xp_x^2 \\ \vdots \\ xp_x^M \end{bmatrix}.$$

(I) Firstly, one proves

$$\hat{R}(x\vec{p}_x) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

or

$$x\vec{p}_x = \hat{R}^{-1}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = W_x \begin{bmatrix} x \\ x \\ \vdots \\ x \end{bmatrix},$$

where W is defined by (22). A direct computation can obtain this.

(II) Secondly, using (29) and (31), we have

$$\begin{aligned}(\partial_{t_n} - \partial_x \mathbb{H}_n)\mathbb{Q}_m &= (\partial_{t_n} - \partial_x \mathbb{H}_n)\hat{R}^m(x\vec{p}_x)\\ &= \hat{R}^m(\partial_{t_n} - \partial_x \mathbb{H}_n)(x\vec{p}_x)\\ &= \hat{R}^m\left[x\partial_x(\vec{p}_{t_n} - \mathbb{H}_n\vec{p}_x) - \mathbb{H}_n\vec{p}_x\right]\\ &= -\hat{R}^m(\vec{p}_{t_n}) = -n(n-1)(n-2)\cdots(n-m+1)\vec{p}_{n-m},\end{aligned}$$

which vanishes if $m \geqslant n$ by (27). □

According to the bi-Hamiltonian theory [14, 22], we can also express $\vec{p}_{\tau_{2m+1}}$ as

$$\begin{aligned}\begin{bmatrix} p^1 \\ p^2 \\ \vdots \\ p^M \end{bmatrix}_{\tau_{2m+1}} &= \hat{R}^{2m+1}\begin{bmatrix} xp^1_x \\ xp^2_x \\ \vdots \\ xp^M_x \end{bmatrix}\\ &= (\hat{R})^{2k}J_1\begin{bmatrix} \frac{\partial \hat{h}_{m-k}}{\partial p^1} \\ \frac{\partial \hat{h}_{m-k}}{\partial p^2} \\ \vdots \\ \frac{\partial \hat{h}_{m-k}}{\partial p^M} \end{bmatrix}, \qquad -\infty < k \leqslant m,\end{aligned} \tag{32}$$

where $\hat{h}'_m s$ are Hamiltonian densities, $m = 1, 2 \ldots$, with m indicating the order of derivatives on which they depend, and

$$\hat{h}_0 = -x\left(\sum_{i=1}^{M}\epsilon_i p^i\right) = -xh_1.$$

We notice that the Hamiltonian densities $\hat{h}_m$ for $m \geqslant 1$ are rational functions in derivatives and can be obtained using the method described in [6, p 69]. But the computation is involved and we do not go further here. One also remarks that $\hat{h}_0$ is *not* the conserved density of (13).

From the theorem, one knows that $\vec{p}_{\tau_{2m+1}}$ commutes with $\vec{p}_{t_n}$ provided $2m+1 \geqslant n$. Inspired by [23] (see also [24, 25]), we have the following

Conjecture. *$\hat{h}_m$ are conserved densities of $\vec{p}_{t_n}$ provided $2m + 1 \geqslant n$.*

In particular, when $n = 2$, we get that for all $\hat{h}_m$, $m = 1, 2, \ldots$, they are conserved densities of (13).

Remark. The Riemann invariants of (13) are

$$\lambda_i = \lambda(u_i),$$

where

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}z}\right|_{z=u_i} = 0,$$

and the associated Lame coefficients Ξ_i are defined by

$$\Xi_i^2(\vec{\lambda}) = \mathrm{Res}_{u_i}\frac{(\mathrm{d}z)^2}{\mathrm{d}\lambda},$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_M)$. Then it is shown that (13) has the conserved density [25]

$$h(\vec{\lambda}, \vec{\lambda}_x) = \sum_{k=1}^{M} \frac{\Xi_i^2}{\lambda_{i,x}}.$$

One can believe the following identity

$$\hat{h}_1 = h(\vec{\lambda}, \vec{\lambda}_x),$$

(up to some scaling) but a proof is still unknown.

4. Concluding remarks

We establish the bi-Hamiltonian structure (26) (or (24)) of the waterbag model (11) using the free energy (17) (or theorem 2.1) of topological field theory associated with it. It turns out that the bi-Hamiltonian structures consist of first-order and third-order Hamiltonian operators when compared with the compatible Dubrovon–Novikov Hamiltonian operators [7, 8]. The reason is that the waterbag model (11) has no Euler vector field or the free energy (17) has no quasi-homogeneous property. In this situation, it is shown that there are compatible Hamiltonian operators of first order and third order using such free energy [16, 17, 19]. On the other hand, since the recursion operator (25) is local, it is natural to think about the rational symmetries of waterbag, which tries to generalize the results in [23] to the n-component case. We conjecture that $\hat{h}_m$ $(m \geqslant 1)$ is a conserved density of the quasi-rational function for the hierarchy under the constraint $2m+1 \geqslant n$. These conserved densities are related to the degenerate Lagrangian representations in flat coordinates for the hierarchy of the waterbag model (see [15, 21]). In Riemann's invariants, these Lagrangian representations are investigated in [19].

One believes that these results can be generalized to the waterbag model of dToda [2]. But the computation is more involved and should be addressed elsewhere.

Acknowledgments

The author is grateful to Professor Leonid V Bogdanov's suggestions and Professor Maxim V Pavlov for his useful discussions on the paper [19]. He also thanks the organizers of the Workshop 'Mathematical Analysis' (July 2006) in National Tai-Tung University for their warm hospitality, where part of the work is finished. The work is supported by the National Science Council for the Taiwan–Russia joint project under grant no NSC 95-2923-M-606-001-MY3.

References

[1] Aoyama S and Kodama Y 1996 Topological Landau–Ginzburg theory with a rational potential and the dKP hierarchy *Commun. Math. Phys.* **182** 185–220

[2] Bogdanov L V and Konopelchenko B G 2004 Symmetry constrains for dispersionless integrable equations and systems of hydrodynamic type *Phys. Lett.* A **330** 448-459 (*Preprint* nlin.SI/0312013) 2003

[3] Boyarsky A, Marshakov A, Ruchayckiy O, Wiegmann P and Zabrodin A 2001 On Associativity equation in dispersionless integrable hierarchies *Phys. Lett.* B **515** 483–92

[4] Carroll R and Kodama Y 1995 Solution of the dispersionless Hirota equations *J. Phys. A: Math. Gen.* **28** 6373–87

[5] Chang J-H 2006 On the water-bag model of dispersionless KP hierarchy *J. Phys. A: Math. Gen.* **39** 11217–30

[6] Dorfman I 1993 *Dirac Structure and Integrability of Nonlinear Evolution Equation* (England: Wiley)

[7] Dubrovin B A and Novikov S P 1983 Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method *Sov. Math. Dokl.* **27** 665–9

[8] Dubrovin B 1996 Geometry of 2D topological field theories *Integrable Systems and Quantum Groups (Springer Lecture Notes in Math.* vol 1620) p 120–348

[9] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627–89

[10] Ferguson J T and Strachan I A B 2006 Logarithmic deformations of the rational superpotential/Landau–Ginzburg construction of solutions of the WDVV equations *Preprint* math-ph/0605078

[11] Kodama Y and Gibbons J 1990 Integrability of dispersionless KP hierarchy *Proc. 4th Workshop on Nonlinear and Turbulent Process in Physics* (Singapore: World Scientific) p 166–80

[12] Kodoma Y 1988 A solution method for dKP Equation *Prog. Theor. Phys. Suppl.* **94** 184–94

[13] Krichever I 1994 The τ-function of the universal Whitham hierarchy, matrix models and Topological Field Theories *Commun. Pure Appl. Math.* **47** 437–75

[14] Magri F 1978 A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62

[15] Mokhov I O 1998 Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems *Usp. Mat. Nauk* **53** 85–V192

Mokhov I O 1998 *Russ. Math. Surveys.* **53** 515–V622 (Engl. Transl.)

[16] Mokhov I O 2003 Quasi-Frobenius algebras and their integrable N-parametric deformations generated by compatible (N x N)-metrics of constant Riemannian curvature *Theor. Math. Phys.* **136** 908–16

[17] Mokhov I O 2004 Nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, integrable hierarchies and the equations of associativity *Preprint* math.DG/0406292

[18] Pavlov M V 2006 Explicit solutions of the WDVV equation determined by the ‘flat’ hydrodynamic reductions of the Egorov hydrodynamic chains *Preprint* nlin.SI/0606008

[19] Pavlov M V 2006 New Hamiltonian formalism and Lagrangian representations for integrable hydrodynamic type systems *Preprint* nlin.SI/0608029

[20] Luen-Chau Li 1999 Classical r-matrix and compatible Possion structures for Lax equations on Possion algebras *Commun. Math. Phys.* **203** 573–92

[21] Nutku Y and Pavlov M V 2002 Multi-Lagrangians for integrable systems *J. Math. Phys.* **43** 1441–59

[22] OLver P J 1993 *Applications of Lie-Groups to Differential Equations* (New York: Springer)

[23] Olver P J and Nutku Y 1988 Hamiltonian structures for systems of hyperbolic conservation laws *J. Math. Phys.* **29** 1610–19

[24] Sheftel M B 2001 Group analysis of hydrodynamic-type systems *Preprint* math-th/0104014

[25] Tsarev S P 1991 The geometry of hamiltonian systems of hydrodynamic type. The generalized hodograph method *Math. USSR Izv.* **37** 397–419

[26] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit *Rev. Math. Phys.* **7** 743